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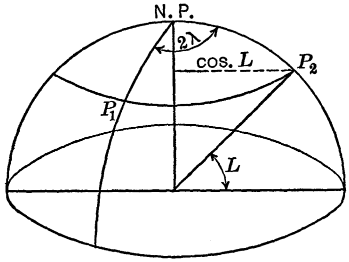

Dropping a perpendicular from the north pole to the great circle through the two points, we have two right triangles. Whence, by Napier's rules of circular parts,

$$\sin D = \sin \lambda \cos L. \tag{2}$$

From (1) and (2), $y = \lambda \cos L - \text{arc} \sin (\sin \lambda \cos L)$, and it is desired to find the maximum value of y.

Let

$$\frac{dy}{dL} = -\lambda \sin L + \frac{\sin L \sin h}{\sqrt{1 - \sin^2 \lambda \cos^2 L}} = 0.$$

Then, $\sin L = 0$ is a solution, giving $L = 0$, obviously the minimum value of y.

The other solution is

$$\cos L = \pm \frac{\sqrt{\lambda^2 - \sin^2 \lambda}}{\lambda \sin \lambda},$$

the maximum. Whence, L can be found for any given λ. As λ varies from 0° to 90°, 2λ varies from 0° to 180° and L varies from 90° to 39° 32′ 23″. For $\lambda = 30°$, $L = 53° \, 34' \, 42''$ and for $\lambda = 60°$, $L = 49° \, 31' \, 16''$.

Also solved by J. A. CAPARO, PAUL CAPRON, HORACE OLSON, C. N. SCHMALL.

382. Proposed by B. J. BROWN, Student in Drury College.

Discuss for what values of m and n, the integral, $\int_0^1 x^{m-1}(1 - x)^{n-1}dx$, is finite and show how this integral can be expressed by means of integrals of the form $\int_0^\infty e^{-x}x^{p-1}dx$.

SOLUTION BY H. L. AGARD, Williams College.

1°. The integrand has no singular points for $m \geqq 1$ and $n \geqq 1$. The only singular points are $x = 0$ for $m < 1$ and $x = 1$ for $n < 1$.

Pierpont[1] states the

THEOREM. Let $f(x)$ be regular in the interval (a, b), except at a. For some $0 < \mu < 1$, let $R \lim_{x=a} (x - a)^\mu |f(x)|$ be finite. Then $f(x)$ is absolutely integrable in (a, b).

[The symbol $R \lim_{x=a}$ indicates that $x \doteq a$ from the right. A similar meaning attaches to $L \lim_{x=a}$. The theorem applies equally to a singularity at b.]

Consider

$$R \lim_{x=0} x^\mu |x^{m-1}(1 - x)^{n-1}|, \quad m < 1.$$

Evidently, this limit is finite for $m > 0$ and $1 - m < \mu < 1$.

Consider

$$L \lim_{x=1} (1 - x)^\mu |x^{m-1}(1 - x)^{n-1}|, \quad n < 1.$$

Evidently, this limit is finite for $n > 0$ and $1 - n < \mu < 1$.

Hence, by the above theorem, the integral is absolutely convergent if both $m > 0$ and $n > 0$.

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[1] Pierpont, *Theory of Functions of Real Variables*, Vol. I, p. 407.

2°. The integral under consideration is known as the Beta function, or first Eulerian integral, and is denoted by $B(m, n)$. This function is to be expressed in terms of the Gamma function, or second Eulerian integral, $\Gamma(m) = \int_0^\infty e^{-x}x^{m-1}dx$. The required relation is

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

In

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \tag{1}$$

set $x = y/(1+y)$, obtaining,

$$B(m, n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}dy. \tag{2}$$

In

$$\Gamma(m) = \int_0^\infty e^{-z}z^{m-1}dz,$$

if we set $z = ax$, we get the formula,

$$\frac{1}{a^m} = \frac{1}{\Gamma(m)}\int_0^\infty e^{-ax}x^{m-1}dx. \tag{3}$$

In (3) replace a by $1 + y$ and m by $m + n$, obtaining,

$$\frac{1}{(1+y)^{m+n}} = \frac{1}{\Gamma(m+n)}\int_0^\infty e^{-(1+y)x}x^{m+n-1}dx. \tag{4}$$

(2) and (4) give

$$B(m, n) = \frac{1}{\Gamma(m+n)}\int_0^\infty dy\int_0^\infty e^{-(1+y)x}x^{m+n-1}y^{m-1}dx. \tag{5}$$

In (5) the order of integration may be inverted.[2] For, (1) the integrand, $f(x, y)$, is continuous in the first quadrant, except on the axes; (2) the integrals, $\int_0^\infty f(x, y)dx$ and $\int_0^\infty f(x, y)dy$ are uniformly convergent[3] in any intervals (a, b), $a > 0$, and (α, β), $\alpha > 0$, respectively; and (3) $\int_0^\infty dx\int_0^\infty f(x, y)dy$ exists, since

$$\begin{aligned}\int_0^\infty f(x, y)dy &= e^{-x}x^{m+n-1}\int_0^\infty e^{-xy}y^{m-1}dy\\ &= e^{-x}x^{m+n-1}\cdot x^{-m}\Gamma(m), \text{ by (3)}\\ &= e^{-x}x^{n-1}\Gamma(m)\end{aligned} \tag{6}$$

and, therefore, for $a > 0$, using (6),

$$\int_0^\infty dx\int_0^\infty f(x, y)dy = \Gamma(m)\lim_{a=0}\int_a^\infty e^{-x}x^{n-1}dx = \Gamma(m)\Gamma(n). \tag{7}$$

The conditions for inversion being fulfilled, (5) and (7) give the desired result,

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

383. Proposed by WILLIAM CULLUM, Albion, Mich.

Find the area of the curved surface of a right cone whose base is the asteroid, $x^{2/3} + y^{2/3} = a^{2/3}$, and whose altitude is h. From Townsend and Goodenough's *First Course in Calculus*, p. 288, Ex. 11.

Note. Among other methods, find the required area by means of the formula

$$\iint\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dydx.$$

EDITORS.

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[2] See Nielson's *Handbuch d. Theorie d. Gamma Functionen*, pages 148–9. Pierpont, *loc. cit.*, § 680, 3.

[3] Pierpont, *loc. cit.*, § 663.